Lecture 1 : Inverse functions

One-to-one Functions A function f is one-to-one if it never takes the same value twice or

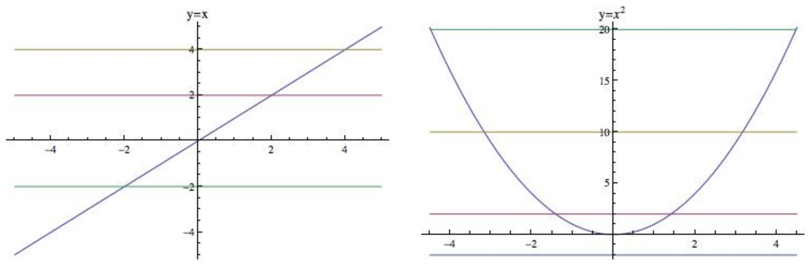
$$\boxed{f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.}$$

Example The function $f(x) = x$ is one to one, because if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

On the other hand the function $g(x) = x^2$ is not a one-to-one function, because $g(-1) = g(1)$.

Graph of a one-to-one function If f is a one to one function then no two points (x_1, y_1), (x_2, y_2) have the same y-value. Therefore no horizontal line cuts the graph of the equation $y = f(x)$ more than once.

Example Compare the graphs of the above functions

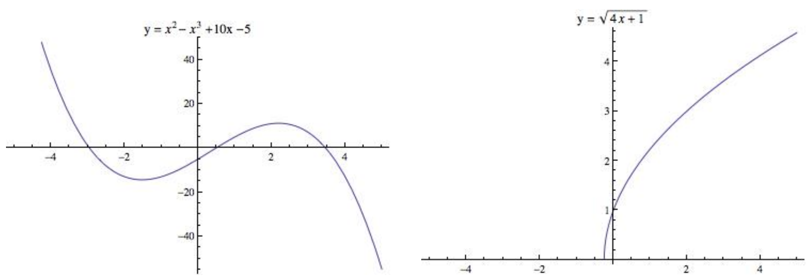


Determining if a function is one-to-one

Horizontal Line test: A graph passes the Horizontal line test if each horizontal line cuts the graph at most once.

Using the graph to determine if f is one-to-one
A function f is one-to-one if and only if the graph $y = f(x)$ passes the Horizontal Line Test.

Example Which of the following functions are one-to-one?



Using the derivative to determine if f is one-to-one
A continuous function whose derivative is always positive or always negative is a one-to-one function. Why?

Example Is the function $g(x) = \sqrt{4x + 4}$ a one-to-one function?

Inverse functions

Inverse Functions If f is a one-to-one function with domain A and range B, we can define an inverse function f^{-1} (with domain B) by the rule

$$\boxed{f^{-1}(y) = x \text{ if and only if } f(x) = y.}$$

This is a sound definition of a function, precisely because each value of y in the domain of f^{-1} has exactly one x in A associated to it by the rule $y = f(x)$.

Example If $f(x) = x^3 + 1$, use the equivalence of equations given above find $f^{-1}(9)$ and $f^{-1}(28)$.

Note that the domain of f^{-1} equals the range of f and the range of f^{-1} equals the domain of f.

Example Let $g(x) = \sqrt{4x+4}$. What is Domain g?
What is Range g?

Does g^{-1} exist?

What is Domain g^{-1}?

What is Range g^{-1}?

What is $g^{-1}(4)$?

Finding a Formula For $f^{-1}(x)$

Given a formula for $f(x)$, we would like to find a formula for $f^{-1}(x)$. Using the equivalence

$$x = f^{-1}(y) \text{ if and only if } y = f(x)$$

we can sometimes find a formula for f^{-1} using the following **method:**

1. In the equation $y = f(x)$, if possible solve for x in terms of y to get a formula $x = f^{-1}(y)$.
2. Switch the roles of x and y to get a formula for f^{-1} of the form $y = f^{-1}(x)$.

Example Let $f(x) = \frac{2x+1}{x-3}$, find a formula for $f^{-1}(x)$.

Composing f and f^{-1}.

We have

$$\text{if} \quad x = f^{-1}(y) \quad \text{then} \quad y = f(x).$$

Substituting $f(x)$ for y in the equation on the left, we get

$$\boxed{f^{-1}(f(x)) = x.}$$

Similarly

$$\text{if} \quad x = f(y) \quad \text{then} \quad y = f^{-1}(x)$$

and substituting $f^{-1}(x)$ for y in the equation on the left, we get

$$\boxed{f(f^{-1}(x)) = x.}$$

Example Above, we found that if $f(x) = \frac{2x+1}{x-3}$, then $f^{-1}(x) = \frac{3x+1}{x-2}$. We can check the above formula for the composition:

$$f(f^{-1}(x)) = f\Big(\frac{3x+1}{x-2}\Big) = \frac{2\Big(\frac{3x+1}{x-2}\Big)+1}{\Big(\frac{3x+1}{x-2}\Big)-3} = \frac{(6x+2+x-2)/(x-2)}{(3x+1-3x+6)/(x-2)} = \frac{7x}{7} = x.$$

You should also check that $f^{-1}(f(x)) = x$.

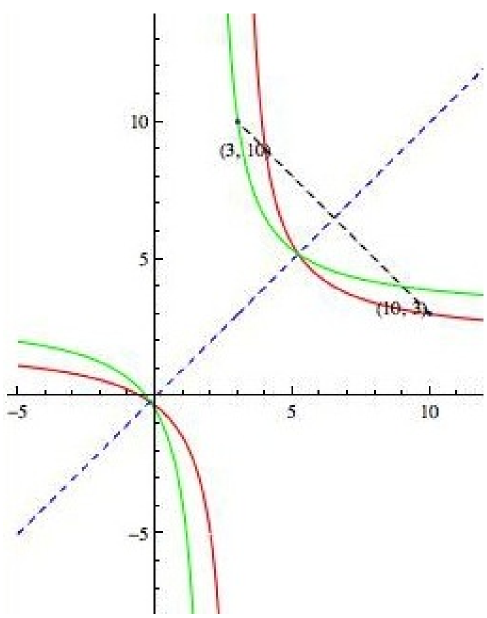
Graph of $y = f^{-1}(x)$

Since the equation $y = f^{-1}(x)$ is the same as the equation $x = f(y)$, the graphs of both equations are identical. To graph the equation $x = f(y)$, we note that this equation results from switching the roles of x and y in the equation $y = f(x)$. This transformation of the equation results in a transformation of the graph amounting to reflection in the line $y = x$. Thus

$$\boxed{\text{the graph of } y = f^{-1}(x) \text{ is a reflection of the graph of } y = f(x) \text{ in the line } y = x \text{ and vice versa.}}$$

Note The reflection of the point (x_1, y_1) n the line $y = x$ is (y_1, x_1). Therefore if the point (x_1, y_1) is on the graph of $y = f^{-1}(x)$, we must have (y_1, x_1) on the graph of $y = f(x)$.

The graphs of $f(x) = \frac{2x+1}{x-3}$ and $f^{-1}(x) = \frac{3x+1}{x-2}$ are shown below.



We can derive properties of the graph of $y = f^{-1}(x)$ from properties of the graph of $y = f(x)$, since they are refections of each other in the line $y = x$. For example:

Theorem If f is a one-to-one continuous function defined on an interval, then its inverse f^{-1} is also one-to-one and continuous. (Thus $f^{-1}(x)$ has an inverse, which has to be $f(x)$, by the equivalence of equations given in the definition of the inverse function.)

Theorem If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

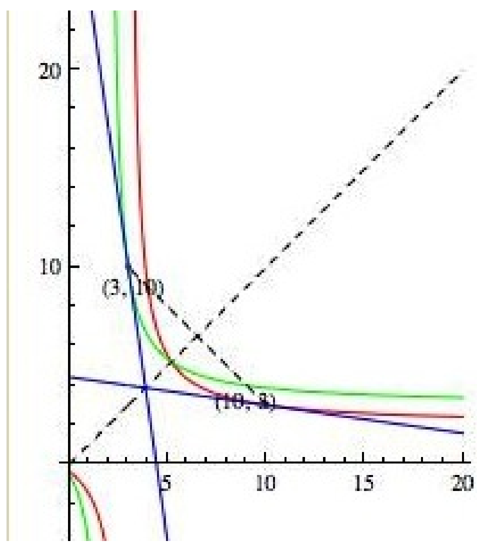
$$\boxed{(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.}$$

proof $y = f^{-1}(x)$ if and only if $x = f(y)$. Using implicit differentiation we differentiate $x = f(y)$ with respect to x to get

$$1 = f'(y)\frac{dy}{dx} \quad \text{or} \quad \frac{1}{f'(y)} = \frac{dy}{dx}$$

$$\text{or} \quad \frac{1}{f'(y)} = (f^{-1})'(x) \quad \text{or} \quad \frac{1}{f'(f^{-1}(x))} = (f^{-1})'(x)$$

Geometrically this means that if $(a, f^{-1}(a))$ is a point on the curve $y = f^{-1}(x)$, then the point $(f^{-1}(a), a)$ is on the curve $y = f(x)$ and the slope of the tangent to the curve $y = f^{-1}(x)$ at $(a, f^{-1}(a))$ is the reciprocal of the tangent to the curve $y = f(x)$ at the point $(f^{-1}(a), a)$. The graphs of the function $f(x) = \frac{2x+1}{x-3}$ and $f^{-1}(x) = \frac{3x+1}{x-2}$ are shown below. You can verify that $-7 = (f^{-1})'(3) = \frac{1}{f'(10)}$.



Note To use the above formula for $(f^{-1})'(a)$, you do not need the formula for $f^{-1}(x)$, you only need the value of f^{-1} at a and the value of f at $f^{-1}(a)$.

Example Consider the function $f(x) = \sqrt{4x+4}$ defined above. Find $(f^{-1})'(4)$.

What does the formula from the theorem say?

Use the equivalence of the equations $y = f^{-1}(x)$ and $x = f(y)$ to find $f^{-1}(4)$.

Put this in the formula from the theorem to find $(f^{-1})'(4)$.

Example Let $f(x) = x^3 + 1$, find $(f^{-1})'(28)$.

Example If f is a one-to-one function with the following properties:

$$f(10) = 21, \quad f'(10) = 2, \quad f^{-1}(10) = 4.5, \quad f'(4.5) = 3.$$

Find $(f^{-1})'(10)$.